

FORCING POSITIVE PARTITION RELATIONS

BY

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ABSTRACT. We show how to force two strong positive partition relations on ω_1 and use them in considering several well-known open problems.

In [32] Sierpiński proved that the well-known Ramsey Theorem [27] does not generalize to the first uncountable cardinal by constructing a partition $[\omega_1]^2 = K_0 \cup K_1$ with no uncountable homogeneous sets. Sierpiński's partition has been analyzed in several directions. One direction was to improve this relation so as to get much stronger negative partition relations on ω_1 . The direction taken in this paper is to prove stronger and stronger positive relations on ω_1 which do not appear to be refutable by Sierpiński's partition. The first result of this kind is due to Dushnik and Miller [9] who proved

$$\omega_1 \rightarrow (\omega_1, \omega)^2.$$

This was later improved by Erdős and Rado [11] to

$$\omega_1 \rightarrow (\omega_1, \omega + 1)^2.$$

In [17] Hajnal proved the following result which shows that the Erdős-Rado theorem is, in a sense, a best possible result of this sort in ZFC:

$$\text{CH implies } \omega_1 \nrightarrow (\omega_1, \omega + 2)^2.$$

Problem 8 of Erdős and Hajnal [12, 13] asks whether $\omega_1 \nrightarrow (\omega_1, \omega + 2)^2$ can be proved without the continuum hypothesis, i.e., whether $\omega_1 \rightarrow (\omega_1, \omega + 2)^2$ is consistent with ZFC. The first result on this problem is due to Laver [24] who proved that

$$\text{MA}_{\aleph_1} \text{ implies } \omega_1 \rightarrow \left(\omega_1, \binom{\omega_1}{\omega} \right)^2.$$

This result was improved by Hajnal (see [24]) to

$$\text{MA}_{\aleph_1} \text{ implies } \omega_1 \rightarrow \left(\omega_1, \binom{\omega_1}{\alpha} \right)^2 \text{ for all } \alpha < \omega_1.$$

Clearly these results leave open the problem whether $\omega_1 \rightarrow (\omega_1, \omega + 2)^2$ is consistent. In this paper we shall prove the consistency of

$$\omega_1 \rightarrow (\omega_1, \alpha)^2 \text{ for all } \alpha < \omega_1.$$

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Let us now consider the following relation introduced by Fred Galvin (it can be considered as a dual of the usual \rightarrow relation). Let ϕ and ψ be order types and let r and κ be cardinals. Then the symbol

$$\phi \overset{*}{\rightarrow} (\psi)_{<\kappa}^r$$

means: If $\phi = \text{tp } A$, and if $[A]^r = \bigcup_{i \in I} K_i$ is a disjoint partition such that $|K_i| < \kappa$ for all $i \in I$, then there is a $B \subseteq A$ such that $\text{tp } B = \psi$ and $|[B]^r \cap K_i| \leq 1$ for all $i \in I$. Let $\phi \overset{*}{\rightarrow} (\psi)_\kappa^r$ iff $\phi \overset{*}{\rightarrow} (\psi)_{<\kappa^+}^r$.

It is easily seen that

$$\phi \rightarrow (\psi)_\kappa^r \text{ implies } \phi \overset{*}{\rightarrow} (\psi)_\kappa^r.$$

Hence, $\omega_1 \overset{*}{\rightarrow} (\omega_1)_2^2$ is a weakening of $\omega_1 \rightarrow (\omega_1)_2^2$, and it is not “obviously” refuted by Sierpiński’s partition. However, Galvin (unpublished) proved that

$$\text{CH implies } \omega_1 \overset{*}{\rightarrow} (\omega_1)_2^2.$$

He asked whether $\omega_1 \overset{*}{\rightarrow} (\omega_1)_2^2$ is a theorem of ZFC or not. We answer this question by proving the consistency of

$$\omega_1 \overset{*}{\rightarrow} (\omega_1)_{<\aleph_0}^2,$$

which is, in a sense, best possible since $\omega_1 \not\rightarrow (3)_{\aleph_0}^2$.

The next problem we consider is the well-known S -space problem from general topology [22, 28, 30]. It essentially asks for a strong partition property of ω_1 . To state this problem we need some definitions. A topological space X is hereditarily separable iff every subspace of X has a countable dense subset. X is called hereditarily Lindelöf iff for every family \mathcal{U} of open subsets of X , there is a countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\bigcup \mathcal{U}_0 = \bigcup \mathcal{U}$. The S -space problem asks whether every regular hereditarily separable topological space is hereditarily Lindelöf. A counterexample to this problem is called an S -space. The problem has been intensively studied since the late 1960’s, and its present formulation is due to several mathematicians [22, 28, 30]. The first example of an S -space was constructed by M. E. Rudin [29] using a Suslin tree. Since then a number of constructions have appeared using various assumptions such as \diamond , CH, ... Also a number of partial nonexistence results have appeared using mainly $\text{MA} + \neg\text{CH}$ (see [22, 28, 30]). In this paper we shall prove the consistency of:

Every regular hereditarily separable topological space is hereditarily Lindelöf.

Hence the S -space problem is undecidable on the basis of the usual axioms of set theory. Working independently and somewhat later, J. Baumgartner proved the consistency of $\text{ZFC} +$ “there are no weak-HFD’s”. (HFD’s form an important class of subspaces of $\{0, 1\}^{\aleph_1}$ used by Hajnal and Juhász and others in constructing various sorts of S -spaces [22, 28].)

Next we are going to consider the problem of bounds on the cardinalities of Hausdorff spaces with no uncountable discrete subspaces. (A set $D \subseteq X$ is a discrete subspace of X if for every $d \in D$ there exists an open subset U_d of X such that $D \cap U_d = \{d\}$.) That the cardinality of such a space has a bound was first independently noticed by Isbell [20] (for completely regular spaces), Efimov [10] and de Groot [16]. The bound they found was $2^{2^{\aleph_0}}$. This bound was improved to $2^{2^{\aleph_0}}$ first by de Groot [16] for the class of all regular spaces, and then by Hajnal and Juhász [18] for the class of all Hausdorff spaces. The natural question which remained unanswered is due to de Groot, Efimov and Isbell [12, Problem 77] and asks whether there exists a Hausdorff space of cardinality $(2^{\aleph_0})^+$ with no uncountable discrete subspaces. The first result on this problem is due to Hajnal and Juhász [19] who constructed such a space using a forcing argument. A compact example has since been constructed by Fedorčuk [14] using \diamond . In this note we shall prove the consistency of:

Every Hausdorff space with no uncountable discrete subspaces has cardinality $\leq 2^{\aleph_0}$.

We shall deduce this result from the consistency of the following statement, which is of independent interest:

If X is a Hausdorff space with no uncountable discrete subspace, then every point of X is the intersection of countably many open subsets of X .

The results of this paper were proved while I was visiting the Department of Mathematics at Dartmouth College during the academic year 1980–81. I would like to express my gratitude to Professor James Baumgartner for making this visit possible. I would also like to thank Professor Fred Galvin for a very stimulating correspondence concerning a class of problems about strong partition relations on ω_1 , a small part of which is considered in this paper. The results of this paper were announced in [34, 35, and 36].

1. In this section we construct a model of $\text{ZFC} + \text{MA}_{\aleph_1}$ in which ω_1 satisfies two strong partition relations which will be used in the rest of the paper. Our forcing terminology is standard (see [5, 21, 23]). All undefined terms concerning the partition calculus can be found in [37]. If $A, B \subseteq \omega_1$, then by $A \otimes B$ we denote $\{\{\alpha, \beta\} : \alpha \in A, \beta \in B, \alpha \neq \beta\}$. If $K \subseteq [\omega_1]^2$ and $\alpha \in \omega_1$, then $K(\alpha)$ denotes the set $\{\beta < \omega_1 : \{\alpha, \beta\} \in K\}$. The symbol

$$\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$$

means: For any partition $[\omega_1]^2 = K_0 \cup K_1$ either

- (1) there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or else
- (2) there is an $A \in [\omega_1]^{\aleph_1}$ and a family \mathfrak{B} of \aleph_1 disjoint finite subsets of ω_1 such that $(\{\alpha\} \otimes F) \cap K_1 \neq \emptyset$ for all $\alpha \in A$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.

The set A which satisfies condition (2) is called a bad set. The purpose of this section is to prove the following theorem.

THEOREM 1. *If ZF is consistent, then so is ZFC plus the following statements simultaneously:*

- (i) $MA + 2^{\aleph_0} = \aleph_2$,
- (ii) $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$,
- (iii) $\omega_1 \xrightarrow{*} (\omega_1)_{<\aleph_0}^2$.

If A is a set, then \mathcal{C}_A denotes the set of all finite partial functions from A into 2 ordered by \supseteq . Thus \mathcal{C}_{ω_1} is the standard poset for adding \aleph_1 Cohen reals. If $\alpha < \beta \leq \omega_1$, then we let

$$\mathcal{C}_{\alpha, \beta} = \{p \in \mathcal{C}_{\omega_1} : p \subseteq [\alpha, \beta)\}.$$

Let $\mathcal{C}_\beta = \mathcal{C}_{0, \beta}$.

Let \mathfrak{E} denote the set of all pairs $\langle a, A \rangle$ where a is a countable closed subset of ω_1 and A is a closed and unbounded subset of ω_1 . We order \mathfrak{E} by

$$\langle a, A \rangle \leq \langle b, B \rangle \quad \text{iff} \quad b = a \cap (\max(b) + 1) \text{ \& } A \subseteq B \text{ \& } a \setminus b \subseteq B.$$

Then \mathfrak{E} is the Jensen closed unbounded set poset [8]. It is clear that \mathfrak{E} is a σ -closed poset. Moreover, every countable set $\mathfrak{E}_0 \subseteq \mathfrak{E}$ of pairwise compatible elements has a greatest lower bound $\langle a, A \rangle \in \mathfrak{E}$ defined by

$$a = \overline{\bigcup \{b : \exists B (\langle b, B \rangle \in \mathfrak{E}_0)\}} \quad \text{and} \quad A = \bigcap \{B : \exists b (\langle b, B \rangle \in \mathfrak{E}_0)\}.$$

Let $G_\mathfrak{E}$ be a generic subset of \mathfrak{E} . Then

$$C_\mathfrak{E} = \bigcup \{a : \exists A (\langle a, A \rangle \in G_\mathfrak{E})\}$$

is a closed unbounded subset of ω_1 which is almost included in every club subset of ω_1 from the ground model [8].

LEMMA 1. *Let $\mathcal{C} = \mathcal{C}_{\omega_1}$ be the standard poset for adding \aleph_1 Cohen reals. Let $G_\mathcal{C}$ be a V -generic subset of \mathcal{C} . Let \mathfrak{E} be the Jensen club set poset in $V[G_\mathcal{C}]$. Let (in V) \mathfrak{Q} be a c.c.c. poset and let $G_\mathfrak{Q}$ be a $V[G_\mathcal{C}]$ -generic subset of \mathfrak{Q} . Let, in $V[G_\mathfrak{Q}]$, $[\omega_1]^2 = K_0 \cup K_1$ and $[\omega_1]^2 = \bigcup_{\xi < \omega_1} J_\xi$ be two disjoint partitions such that the first partition has no bad sets, while each color of the second partition is finite. Let $G_\mathfrak{E}$ be a $V[G_\mathcal{C}][G_\mathfrak{Q}]$ -generic subset of \mathfrak{E} . In $V[G_\mathcal{C}][G_\mathfrak{Q}][G_\mathfrak{E}]$, we define*

$$\begin{aligned} \mathfrak{S}_0 &= \{s \in [\omega_1]^{<\aleph_0} : s \text{ is separated by } C_\mathfrak{E} \text{ \& } [s]^2 \subseteq K_0\}, \\ \mathfrak{S}_1 &= \{s \in [\omega_1]^{<\aleph_0} : s \text{ is separated by } C_\mathfrak{E} \text{ \& } \forall \xi < \omega_1 [s]^2 \cap J_\xi \neq \emptyset\}. \end{aligned}$$

Let \mathfrak{S}_0 and \mathfrak{S}_1 be partially ordered by \supseteq . Then both \mathfrak{S}_0 and \mathfrak{S}_1 are c.c.c. posets in $V[G_\mathcal{C}][G_\mathfrak{Q}][G_\mathfrak{E}]$.

PROOF. We first prove the lemma for the poset \mathfrak{S}_0 . So let, in $V[G_\mathcal{C}][G_\mathfrak{Q}][G_\mathfrak{E}]$, $\langle s_\xi : \xi < \omega_1 \rangle$ be an ω_1 -sequence of elements of \mathfrak{S}_0 . By the standard Δ -system argument we may assume that s_ξ 's are disjoint, increasing and of the same cardinality n , where $1 \leq n < \omega$.

From now on we work in $V[G_{\mathcal{C}}][G_2]$ and fix an \mathcal{E} -name $\langle \dot{s}_\xi: \xi < \omega_1 \rangle$ for the sequence $\langle s_\xi: \xi < \omega_1 \rangle$ and a condition $\langle a_0, A_0 \rangle \in \mathcal{E}$ which forces that n and $\langle \dot{s}_\xi: \xi < \omega_1 \rangle$ have the above properties. We shall find $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ such that

$$\langle b, B \rangle \Vdash_{\mathcal{E}} \exists \xi < \eta < \omega_1 (\dot{s}_\xi \otimes \dot{s}_\eta \subseteq K_0).$$

This will finish the proof of Lemma 1 for the poset \mathcal{S}_0 .

Note that \mathcal{E} is defined in $V[G_{\mathcal{C}}]$, hence it is not necessarily σ -closed in $V[G_{\mathcal{C}}][G_2]$. Instead of \mathcal{E} we shall work with the set of all $\langle a, A \rangle \in \mathcal{E}$ such that $A \in V$. The restriction causes no loss of generality since this set is dense in \mathcal{E} . By an abuse of notation we denote it also by \mathcal{E} .

Let θ be a big enough regular cardinal, and let N_0 be a countable elementary submodel of H_θ such that $N_0 \cap V[G_{\mathcal{C}}] \in V[G_{\mathcal{C}}]$, and such that \mathcal{E} , $\langle a_0, A_0 \rangle$, $\langle \dot{s}_\xi: \xi < \omega_1 \rangle$, K_0 , $K_1 \in N_0$. Since $V[G_{\mathcal{C}}][G_2]$ is a c.c.c. extension of $V[G_{\mathcal{C}}]$, such a submodel exists. Let $\delta_0 = N_0 \cap \omega_1$, and let $F \subseteq \omega_1 \setminus \delta_0$ be a fixed set of size n . Let

$$\mathcal{W}_F = \left\{ \langle a, A \rangle \in \mathcal{E} \cap N_0 : (\langle a, A \rangle \perp \langle a_0, A_0 \rangle) \vee \left(\langle a, A \rangle \leq \langle a_0, A_0 \rangle \& \exists s \subseteq \delta_0 \exists \xi < \delta_0 (s \otimes F \subseteq K_0 \& \langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = s) \right) \right\}.$$

CLAIM 1. \mathcal{W}_F is a dense open subset of $\mathcal{E} \cap N_0$.

PROOF. Let $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ be a given element of $\mathcal{E} \cap N_0$. By induction on $0 \leq i \leq n$, for each strictly increasing sequence $\langle x_1, \dots, x_{n-i} \rangle$ of ordinals $< \omega_1$, we define the statements $\Phi_{n-i}(x_1, \dots, x_{n-i})$ as follows:

$$\Phi_n(x_1, \dots, x_n) \quad \text{iff } \exists \xi < \omega_1 \exists \langle a, A \rangle \leq \langle b, B \rangle (\langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = \{x_1, \dots, x_n\});$$

$$\Phi_{n-i}(x_1, \dots, x_{n-i}) \quad \text{iff } |\{y < \omega_1 : \Phi_{n-i+1}(x_1, \dots, x_{n-i}, y)\}| = \aleph_1 \quad \text{for } 0 < i \leq n.$$

We shall prove that Φ_0 holds.

Starting from N_0 we build a strictly increasing, continuous sequence $\langle N_\alpha: \alpha < \omega_1 \rangle$ of countable elementary submodels of H_θ . Let $\delta_\alpha = N_\alpha \cap \omega_1$ for $\alpha < \omega_1$, and let $D = \{\delta_\alpha: \alpha < \omega_1\}$. Then D is a closed unbounded subset of ω_1 . Since $\mathcal{C} \times \mathcal{Q}$ is a c.c.c. poset, we can find $\langle a', A' \rangle \leq \langle b, B \rangle$ and $\gamma < \omega_1$ such that $A' \setminus \gamma \subseteq D$. Choose $\langle a, A \rangle \leq \langle a', A' \rangle$, $\xi < \omega_1$, and $\{z_1, \dots, z_n\} \subseteq \omega_1 \setminus \gamma$ such that

$$\langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = \{z_1, \dots, z_n\}.$$

This shows that $\Phi_n(z_1, \dots, z_n)$ holds. By induction on i we shall now show that $\Phi_{n-i}(z_1, \dots, z_{n-i})$ holds for all $0 \leq i < n$. Note that $\{z_1, \dots, z_n\}$ is separated by D , so we can find $\alpha_1 < \dots < \alpha_n < \omega_1$ such that $\delta_{\alpha_i} \leq z_i < \delta_{\alpha_{i+1}}$ for all $i = 1, \dots, n$. So let us assume that $\Phi_{n-i+1}(z_1, \dots, z_{n-i+1})$ holds for some $0 < i < n$. Let

$$Z_{n-i+1} = \{z < \omega_1 : \Phi_{n-i+1}(z_1, \dots, z_{n-i}, z)\}.$$

Then $z_{n-i+1} \in Z_{n-i+1}$ by the assumption. Note that the parameters in the definition of Z_{n-i+1} are all members of $N_{\alpha_{n-i+1}}$, which implies $Z_{n-i+1} \in N_{\alpha_{n-i+1}}$. Hence we must have $N_{\alpha_{n-i+1}} \models Z_{n-i+1}$ is uncountable. Hence Z_{n-i+1} is really uncountable. This shows that $\Phi_{n-i}(z_1, \dots, z_{n-i})$ holds and finishes the induction step.

Thus, in particular, $\Phi_1(z_1)$ holds, and by repeating the above argument we conclude that $\{z < \omega_1 : \Phi_1(z)\}$ is uncountable. Hence Φ_0 holds.

We have already noted that the parameters of each statement Φ_i ($0 \leq i \leq n$) are members of N_0 . Hence, if we let $Y_1 = \{y < \omega_1 : \Phi_1(y)\}$ then $Y_1 \in N_0$, and by the fact that Φ_0 holds, Y_1 is uncountable. We claim that, for some $y \in Y_1 \cap \delta_0$, $\{y\} \otimes F \subseteq K_0$. Otherwise the following holds:

$$N_0 \models \forall \delta < \omega_1 \exists F_\delta \in [\omega_1 \setminus \delta]^\omega \forall y \in Y_1 \cap \delta ((\{y\} \otimes F_\delta) \cap K_1 \neq \emptyset).$$

Since $N_0 < H_\theta$, this sentence also holds in H_θ . However, this easily gives a bad set with respect to $[\omega_1]^2 = K_0 \cup K_1$, contradicting the assumption that $V[G_\mathbb{C}][G_2]$ is a property K extension of $V[G_2]$ which contains no bad sets. So pick $y_1 \in Y_1 \cap \delta_0$ such that $\{y_1\} \otimes F \subseteq K_0$. Let $Y_2 = \{y < \omega_1 : \Phi_2(y_1, y)\}$. Then by the assumption that $\Phi_1(y_1)$ holds, Y_2 is uncountable. Clearly $Y_2 \in N_0$. By repeating the above argument, we can find a $y_2 \in Y_2 \cap \delta_0$ such that $\{y_2\} \otimes F \subseteq K_0$. Proceeding in this way we construct $y_1 < y_2 < \dots < y_n$ such that

$$\{y_1, \dots, y_n\} \otimes F \subseteq K_0, \text{ and } \Phi_n(y_1, \dots, y_n) \text{ holds.}$$

Hence $N_0 \models \Phi(y_1, \dots, y_n)$. This means that we can find $\xi < \delta_0$ and $\langle a, A \rangle \leq \langle b, B \rangle$ such that $\langle a, A \rangle \in \mathcal{E} \cap N_0$ and

$$\langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = \{y_1, \dots, y_n\}.$$

This shows that $\langle a, A \rangle \in \mathcal{W}_F$ and completes the proof of Claim 1.

Define $R \subseteq (\mathcal{E} \cap N_0) \times \delta_0 \times [\delta_0]^n$ by

$$R(\langle a, A \rangle, \xi, s) \text{ iff } \langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = s.$$

Since $\mathcal{E} \cap N_0$ and R can be coded using only a countable amount of information, we can find some $\alpha < \omega_1$ such that

$$\mathcal{E} \cap N_0 \in V[G_{\mathbb{C}_\alpha}] \text{ and } R \in V[G_{\mathbb{C}_\alpha}][G_2].$$

In $V[G_{\mathbb{C}_\alpha}]$ we choose a mapping π which maps $\mathcal{C}_{\alpha, \alpha+\omega}$ isomorphically into a dense subset of $\{\langle a, A \rangle \in \mathcal{E} \cap N_0 : \langle a, A \rangle \leq \langle a_0, A_0 \rangle\}$. Let

$$G = G_{\mathcal{C}_{\alpha, \alpha+\omega}} = G_{\mathbb{C}} \cap \mathcal{C}_{\alpha, \alpha+\omega}.$$

Then G is a $V[G_{\mathbb{C}_\alpha}]$ -generic subset of $\mathcal{C}_{\alpha, \alpha+\omega}$. So $\pi''G$ is a countable pairwise compatible subset of \mathcal{E} . Hence $\pi''G$ has a lower bound $\langle \bar{a}, \bar{A} \rangle$ in \mathcal{E} .

Note that, for each $F \in [\omega_1 \setminus \delta_0]^n$, \mathcal{W}_F is definable from $F, K_0, K_1, \mathcal{E} \cap N_0, \langle a_0, A_0 \rangle, \delta_0$, and R . Hence, for each $F \in [\omega_1 \setminus \delta_0]^n$, $\mathcal{W}_F \in V[G_{\mathbb{C}_\alpha}][G_2]$. Since G is also a $V[G_{\mathbb{C}_\alpha}][G_2]$ -generic subset of $\mathcal{C}_{\alpha, \alpha+\omega}$, it follows that $\pi''G$ intersects each \mathcal{W}_F for $F \in [\omega_1 \setminus \delta_0]^n$. Hence, for each $F \in [\omega_1 \setminus \delta_0]^n$, there is an $\langle a, A \rangle \in \mathcal{W}_F$ such that $\langle \bar{a}, \bar{A} \rangle \leq \langle a, A \rangle$.

Pick $\langle b, B \rangle \leq \langle \bar{a}, \bar{A} \rangle$, $\eta < \omega_1$, and $F \in [\omega_1 \setminus \delta_0]^n$ such that $\langle b, B \rangle \Vdash_{\mathcal{E}} \dot{s}_\eta = F$. Let $\langle a, A \rangle \in \mathcal{W}_F$ be such that $\langle b, B \rangle \leq \langle a, A \rangle$. Thus, for some $s \in [\delta_0]^n$ and $\xi < \delta_0$, we have

$$s \otimes F \subseteq K_0 \text{ and } \langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = s.$$

Hence

$$\langle b, B \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi \otimes \dot{s}_\eta \subseteq K_0.$$

This completes the proof of Lemma 1 for the case of poset \mathcal{S}_0 .

The proof that \mathcal{S}_1 is a c.c.c. poset in $V[G_\mathbb{C}][G_\mathbb{Q}][G_\mathbb{E}]$ is similar, so we mention only the main differences. Again we start with a given sequence $\langle s_\xi: \xi < \omega_1 \rangle$ of elements of \mathcal{S}_1 . We may assume the s_ξ 's form a Δ -system with root s , and if we let $t_\xi = s_\xi \setminus s$ for $\xi < \omega_1$, then the t_ξ 's are strictly increasing and of the same cardinality n , where $1 \leq n < \omega$.

Working in $V[G_\mathbb{C}][G_\mathbb{Q}]$, we fix \mathbb{E} -names $\langle \dot{s}_\xi: \xi < \omega_1 \rangle$ and $\langle \dot{t}_\xi: \xi < \omega_1 \rangle$ for the sequences $\langle s_\xi: \xi < \omega_1 \rangle$ and $\langle t_\xi: \xi < \omega_1 \rangle$, and a condition $\langle a_0, A_0 \rangle \in \mathbb{E}$ which forces that $n, s, \langle \dot{s}_\xi: \xi < \omega_1 \rangle$ and $\langle \dot{t}_\xi: \xi < \omega_1 \rangle$ have the above properties. We need to find $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ such that

$$\langle b, B \rangle \Vdash_{\mathbb{E}} \exists \xi < \eta < \omega_1 \forall \zeta < \omega_1 \left| [\dot{s}_\xi \cup \dot{s}_\eta]^2 \cap J_\zeta \right| \leq 1.$$

We fix a cardinal θ and a countable elementary symbol N_θ of H_θ as before. Let

$$D = \left\{ \delta < \omega_1: \lim(\delta) \ \& \ \forall \alpha < \delta \ \forall \zeta < \omega_1 \left(J_\zeta \cap [\delta]^2 \neq \emptyset \Rightarrow J_\zeta(\alpha) \subseteq \delta \right) \right\}.$$

Then D is a closed unbounded subset of ω_1 such that $D \in N_\theta$. Now we fix an $F \in [\omega_1 \setminus \delta_0]^n$ which is separated by D such that

$$\forall \zeta < \omega_1 \left| [s \cup F]^2 \cap J_\zeta \right| \leq 1,$$

and define

$$\begin{aligned} \mathcal{W}_F = \Big\{ \langle a, A \rangle \in \mathbb{E} \cap N: & \left(\langle a, A \rangle \perp \langle a_0, A_0 \rangle \right) \\ & \vee \left(\langle a, A \rangle \leq \langle a_0, A_0 \rangle \ \& \ \exists t \subseteq \delta_0 \ \exists \xi < \delta_0 \right. \\ & \left. \left(\forall \zeta < \omega_1 \left| [s \cup t \cup F]^2 \cap J_\zeta \right| \leq 1 \ \& \ \langle a, A \rangle \Vdash_{\mathbb{E}} \dot{i}_\xi = t \right) \right) \Big\}. \end{aligned}$$

As before we claim that \mathcal{W}_F is a dense open subset of $\mathbb{E} \cap N_\theta$. So let $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ be a given element of $\mathbb{E} \cap N_\theta$. The statements $\Phi_{n-i}(x_1, \dots, x_{n-i})$ ($0 \leq i \leq n$) are defined as before. The proof that Φ_0 holds is also the same.

Let $Y_1 = \{y < \omega_1: \Phi_1(y)\}$. Then Y_1 is uncountable and $Y_1 \in N_\theta$. Let $\{\zeta_1, \dots, \zeta_{k_1}\}$ be a list of all $\zeta < \omega_1$ such that $[s \cup F]^2 \cap J_\zeta \neq \emptyset$. Since each J_ζ is finite, and since $Y_1 \cap \delta_0$ is infinite, we can find $y_1 \in Y_1 \cap \delta_0$ such that

$$(\{y_1\} \otimes (s \cup F)) \cap J_\zeta = \emptyset \quad \text{for all } \zeta \in \{\zeta_1, \dots, \zeta_{k_1}\}.$$

We claim that

$$\left| [s \cup \{y_1\} \cup F]^2 \cap J_\zeta \right| \leq 1 \quad \text{for all } \zeta < \omega_1.$$

Otherwise, let $\zeta < \omega_1$ be such that $[s \cup \{y_1\} \cup F]^2 \cap J_\zeta$ contains two different edges l_0 and l_1 . It is clear that l_0 and l_1 cannot both be subsets of $s \cup \{y_1\}$ since some condition from \mathbb{E} forces this set to be a subset of a member of \mathcal{S}_1 . From the definition of D it easily follows that $\max l_0 = \max l_1$. Hence, $\max l_0 = \max l_1 \in F$. Hence, for some $i < 2$, $l_i \in s \otimes F$, which means that $\zeta \in \{\zeta_1, \dots, \zeta_{k_1}\}$. It follows that

$$(\{y_1\} \otimes (s \cup F)) \cap J_\zeta = \emptyset.$$

Consequently, $\min l_0 \neq y_1$ and $\min l_1 \neq y_1$, which yields the contradiction $l_1, l_2 \in [s \cup F]^2$.

Now let $Y_2 = \{y < \omega_1 : \Phi_2(y_1, y)\}$. Then Y_2 is uncountable and $Y_2 \in N_0$. Working as above, we can find $y_2 \in Y_2 \cap \delta$ such that

$$\left| [s \cup \{y_1, y_2\} \cup F]^2 \cap J_\xi \right| \leq 1 \quad \text{for all } \xi < \omega_1.$$

Proceeding in this way we construct $y_1 < y_2 < \dots < y_n < \delta_0$ such that

$$\forall \xi < \omega_1 \left| [s \cup \{y_1, \dots, y_n\} \cup F]^2 \cap J_\xi \right| \leq 1,$$

and

$$\Phi_n(y_1, \dots, y_n) \text{ holds.}$$

Hence $N_0 \models \Phi_n(y_1, \dots, y_n)$. So we can find $\xi < \delta_0$ and $\langle a, A \rangle \leq \langle a_0, A_0 \rangle$ such that $\langle a, A \rangle \in \mathcal{S} \cap N_0$ and

$$\langle a, A \rangle \Vdash_{\mathcal{S}} i_\xi = \{y_1, \dots, y_n\}$$

This shows that $\langle a, A \rangle \in \mathcal{W}_F$. Hence, \mathcal{W}_F is a dense open subset of $\mathcal{S} \cap N_0$.

We leave the remainder of the proof of Lemma 1 to the reader since the rest of the proof for \mathcal{S}_1 is like the proof for \mathcal{S}_0 .

Now we are going to describe a mixed iteration $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$ of Cohen and Jensen partially ordered sets. This will be done by induction on α , and for this purpose let E and O denote the sets of all even and odd ordinals $< \omega_2$, respectively.

If $\alpha = 0$, then $\mathcal{P}_\alpha = \emptyset$.

If $\alpha \in E$, then $\mathcal{P}_{\alpha+1}$ is the set of all functions p with domain $\alpha + 1$ such that $p \restriction \alpha \in \mathcal{P}_\alpha$ and

$$\Vdash_{\mathcal{P}_\alpha} p(\alpha) \text{ is a member of } \mathcal{C}_{\{\alpha\} \times \omega_1}.$$

If $p, q \in \mathcal{P}_{\alpha+1}$, let $p \leq q$ iff $p \restriction \alpha \leq q \restriction \alpha$ and $p \restriction \alpha \Vdash_{\mathcal{P}_\alpha} p(\alpha) \supseteq q(\alpha)$.

If $\alpha \in O$, then $\mathcal{P}_{\alpha+1}$ is the set of all functions p with domain $\alpha + 1$ such that $p \restriction \alpha \in \mathcal{P}_\alpha$ and

$$\Vdash_{\mathcal{P}_\alpha} p(\alpha) \text{ is a member of the Jensen club set poset } \mathcal{C}_\alpha.$$

If $p, q \in \mathcal{P}_{\alpha+1}$, let $p \leq q$ iff $p \restriction \alpha \leq q \restriction \alpha$ and $p \restriction \alpha \Vdash_{\mathcal{P}_\alpha} p(\alpha) \leq q(\alpha)$ in the Jensen ordering.

If α is a limit ordinal with $\text{cf } \alpha = \omega$, then \mathcal{P}_α is the set of all functions p with domain α such that $p \restriction \beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$, and for some $\gamma < \alpha$, $\Vdash_{\mathcal{P}_\beta} p(\beta) = \emptyset$ for all $\beta \in [\gamma, \alpha) \cap E$.

If α is a limit ordinal with $\text{cf } \alpha > \omega$, then \mathcal{P}_α is the set of all functions p with domain α such that $p \restriction \beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$, and for some $\gamma < \alpha$, $\Vdash_{\mathcal{P}_\beta} p(\beta) = \emptyset$ for $\beta \in [\gamma, \alpha) \cap E$ and $\Vdash_{\mathcal{P}_\beta} p(\beta) = \langle \emptyset, \omega_1 \rangle$ for $\beta \in [\gamma, \alpha) \cap O$.

In both limit cases we put $p \leq q$ iff $p \restriction \beta \leq q \restriction \beta$ for all $\beta < \alpha$.

From now on let $\alpha \leq \omega_2$ be a fixed ordinal. For $p \in \mathcal{P}_\alpha$ we define $\text{supp}(p) = \{\beta < \alpha : p(\beta) \neq \emptyset \text{ if } \beta \in E \text{ and } p(\beta) \neq \langle \emptyset, \omega_1 \rangle \text{ if } \beta \in O\}$. Then it is easily checked that $\text{supp}(p) \cap E$ is finite, and that $\text{supp}(p) \cap O$ is at most countable.

We say that $p \in \mathcal{P}_\alpha$ is a determined condition if, for every $\beta \in \text{supp}(p) \cap E$, there is some $s_\beta(p) \in \mathcal{C}_{\{\gamma\} \times \omega_1}$ such that $p(\beta) = s_\beta(p)$ (more precisely, $p(\beta) = (s_\beta(p))^\vee$). If $p \in \mathcal{P}_\alpha$ is a determined condition, then by $\sigma(p)$ we denote

$$\cup \{s_\beta(p) : \beta \in \text{supp}(p) \cap E\}$$

considered as a member of $\mathcal{C}(\alpha) := \mathcal{C}_{(E \cap \alpha) \times \omega_1}$. By induction on α it is easily seen that the set of all determined conditions is dense in \mathcal{P}_α . So from now on we shall always work with determined conditions, and use \mathcal{P}_α informally to denote also the set of all determined numbers of \mathcal{P}_α .

Note that the function $\sigma: \mathcal{P}_\alpha \rightarrow \mathcal{C}(\alpha)$ is order preserving, and has the property that if $\sigma(p) = r$ and $r' \leq r$, then for some $p' \leq p$, $\sigma(p') = r'$. Hence, forcing with \mathcal{P}_α can be considered as forcing first with $\mathcal{C}(\alpha)$ and then with $\{p \in \mathcal{P}_\alpha: \sigma(p) \in G_{\mathcal{C}(\alpha)}\}$.

LEMMA 2. *For every $\dot{f} \in V^{\mathcal{P}_\alpha}$ and $p_0 \in \mathcal{P}_\alpha$ with the property $p_0 \Vdash_{\mathcal{P}_\alpha} \dot{f}: \omega \rightarrow On$, there are $\dot{g} \in V^{\mathcal{C}(\alpha)}$ and $p \leq p_0$ such that $\sigma(p) = \sigma(p_0)$ and $p \Vdash_{\mathcal{P}_\alpha} \dot{f} = \dot{g}$.*

PROOF. If $p \in \mathcal{P}_\alpha$ and if $r \in \mathcal{C}(\alpha)$ is compatible with $\sigma(p)$, then $p \wedge r$ denotes the following member q of \mathcal{P}_α . If $\beta \in O$, then $q(\beta) = p(\beta)$. If $\beta \in E$, then $q(\beta) = p(\beta) \cup (r \upharpoonright \{\beta\} \times \omega_1)$. It is clear that this is a well-defined condition and that $p \wedge r \leq p$.

CLAIM 2. Assume $p, p' \in \mathcal{P}_\alpha$ are such that $p' \leq p$. Then there exists a $q \in \mathcal{P}_\alpha$, with $q \leq p$ such that $\sigma(q) = \sigma(p)$, and $p' = q \wedge \sigma(p')$.

PROOF. We define $q \upharpoonright \beta$ by induction on $\beta < \alpha$. Assume $q \upharpoonright \beta$ is defined. If $\beta \in E$, let $q(\beta) = p(\beta)$. If $\beta \in O$, let $q(\beta)$ be a \mathcal{P}_β -name for a member of the Jensen poset $\dot{\mathcal{E}}_\beta$ which is equal to $p'(\beta)$ if $p' \upharpoonright \beta$ is a member of $G_{\mathcal{P}_\beta}$, and equal to $p(\beta)$ otherwise. If β is a limit ordinal, let $q \upharpoonright \beta = \bigcup \{q \upharpoonright \gamma: \gamma < \beta\}$. Now by induction on $\beta < \alpha$ one easily checks that

$$q \upharpoonright \beta \in \mathcal{P}_\beta, \quad q \upharpoonright \beta \leq p \upharpoonright \beta, \quad \sigma(q \upharpoonright \beta) = \sigma(p \upharpoonright \beta),$$

and

$$p' \upharpoonright \beta \leq (q \wedge \sigma(p')) \upharpoonright \beta \leq p' \upharpoonright \beta.$$

This completes the proof of Claim 2.

Starting from p_0 , by induction on $n < \omega$, we define a sequence $\langle p_n: n < \omega \rangle$ of members of \mathcal{P}_α , and for each $n < \omega$ sequences $\langle r_\xi^n: \xi < \delta_n \rangle$ and $\langle x_\xi^n: \xi < \delta_n \rangle$ of members of $\mathcal{C}(\alpha)$ and On , respectively, such that

- (1) $p_{n+1} \leq p_n$,
- (2) $\sigma(p_{n+1}) = \sigma(p_n)$,
- (3) $p_{n+1} \wedge r_\xi^n \Vdash_{\mathcal{P}_\alpha} \dot{f}(n) = x_\xi^n$,
- (4) $\{r_\xi^n: \xi < \delta_n\}$ is a maximal antichain below $\sigma(p_n)$.

Let us first see how to prove the lemma using such sequences. By induction on $\beta \leq \alpha$ we construct $p \upharpoonright \beta \in \mathcal{P}_\beta$ such that $p \upharpoonright \beta \leq p_n \upharpoonright \beta$ and $\sigma(p \upharpoonright \beta) = \sigma(p_n \upharpoonright \beta)$ for all $n < \omega$. Assume $p \upharpoonright \beta$ is constructed. If $\beta \in E$, let $p(\beta) = p_0(\beta)$. If $\beta \in O$, let $p(\beta)$ be a \mathcal{P}_β -name for the greatest lower bound of $\{p_n(\beta): n < \omega\}$ in $\dot{\mathcal{E}}_\beta$. If β is a limit ordinal, let $p \upharpoonright \beta = \bigcup \{p \upharpoonright \gamma: \gamma < \beta\}$. Then it is easily checked that $p \upharpoonright \beta$ is a well-defined condition, and that $p = p \upharpoonright \alpha$ has the properties $p \leq p_n$ and $\sigma(p) = \sigma(p_n)$ for all $n < \omega$. Since p is uniquely determined by $\langle p_n: n < \omega \rangle$, we shall denote it by $\bigwedge_{n < \omega} p_n$.

Define $\dot{g} \in V^{\mathcal{C}(\alpha)}$ to be a function from ω into the ordinals such that

$$\|\dot{g}(n) = x_\xi^n\| = r_\xi^n \quad \text{for } n < \omega \text{ and } \xi < \delta_n.$$

Then by (1)–(4) we have

$$p \Vdash_{\mathcal{P}_\alpha} \dot{f} = \dot{g}.$$

So we are left with the construction of $\langle p_n: n < \omega \rangle$. Assume p_n is defined. By induction on $\xi < \delta_n$ we define sequences $\langle q_\xi^n: \xi < \delta_n \rangle$, $\langle r_\xi^n: \xi < \delta_n \rangle$ of members of \mathcal{P}_α , $\mathcal{C}(\alpha)$ and \mathcal{O}_n , respectively, such that

- (5) $q_\xi^n \leq q_\xi^n \leq p_n$ for $\xi < \zeta$,
- (6) $\sigma(q_\xi^n) = \sigma(p_n)$,
- (7) $r_\xi^n \leq \sigma(p_n)$,
- (8) $r_\xi^n \perp r_\zeta^n$ for $\xi \neq \zeta$,
- (9) $q_\xi^n \wedge r_\xi^n \Vdash_{\mathcal{P}_\alpha} \dot{f}(n) = x_\xi^n$.

The ordinal δ_n is a countable ordinal determined by the fact that $\{r_\xi^n: \xi < \delta_n\}$ is a maximal antichain below $\sigma(p_n)$. Assume q_ξ^n 's, r_ξ^n 's and x_ξ^n 's are defined for every $\xi < \zeta < \omega_1$. If $\{r_\xi^n: \xi < \zeta\}$ is a maximal antichain below $\sigma(p_n)$, we let $\delta_n = \zeta$ and $p_{n+1} = \bigwedge_{\xi < \delta_n} q_\xi^n$. Clearly (1)–(4) are satisfied. So let us assume $\{r_\xi^n: \xi < \zeta\}$ is not a maximal antichain below $\sigma(p_n)$. Pick $r \leq \sigma(p_n)$ which is incompatible with each r_ξ^n ($\xi < \zeta$). Let $q = \bigwedge_{\xi < \zeta} q_\xi^n$. Choose $q' \leq q \wedge r$ and x_ζ^n such that $q' \Vdash_{\mathcal{P}_\alpha} \dot{f}(n) = x_\zeta^n$. By Claim 2 we can find $q_\zeta^n \leq q$ such that $\sigma(q_\zeta^n) = \sigma(q) = \sigma(p_n)$ and $q' = q_\zeta^n \wedge \sigma(q')$. Let $r_\zeta^n = \sigma(q')$. It is clear that (5)–(9) are satisfied. This completes the proof of Lemma 2.

Note that, in particular, Lemma 2 shows that \mathcal{P}_α preserves \aleph_1 . If $\alpha < \omega_2$, let

$$\mathcal{P}_{\alpha, \omega_2} = \{p \upharpoonright (\omega_2 \setminus \alpha): p \in \mathcal{P}_\alpha\}.$$

Let $\mathcal{P}_{\alpha, \omega_2}$ be ordered in $V^{\mathcal{P}_\alpha}$ by the ordering $\dot{\leq}$ defined as follows:

$$q' \dot{\leq} q \text{ iff } \exists p \in G_{\mathcal{P}_\alpha} (p \cup q' \leq p \cup q).$$

Then $\mathcal{P}_{\omega_2} = \mathcal{P}_\alpha * \mathcal{P}_{\alpha, \omega_2}$, and Lemma 2 easily gives

$$\Vdash_{\mathcal{P}_\alpha} \mathcal{P}_{\alpha, \omega_2} \text{ is equivalent to } \dot{\mathcal{P}}_{\omega_2}.$$

LEMMA 3. *Assume CH. Then \mathcal{P}_{ω_2} satisfies the \aleph_2 -chain condition.*

PROOF. Since elements of \mathcal{P}_{ω_2} have countable supports, a standard application of Fodor's Lemma shows that we may restrict ourselves to proving that, for each $\alpha < \omega_2$, \mathcal{P}_α satisfies the \aleph_2 -c.c.

So let $\alpha < \omega_2$ and let $\mathcal{Q}_\alpha \subseteq \mathcal{P}_\alpha$ be the set of all $p \in \mathcal{P}_\alpha$ such that for every $\beta \in \mathcal{O}$, there is a $\mathcal{C}(\beta)$ -name a_p^β for a countable closed subset of ω_1 and a \mathcal{P}_β -name A_p^β for a closed and unbounded subset of ω_1 such that $p \upharpoonright \beta \Vdash_{\mathcal{P}_\beta} p(\beta) = \langle a_p^\beta, A_p^\beta \rangle$.

CLAIM 3. For every $q \in \mathcal{P}_\alpha$ there is a $p \in \mathcal{Q}_\alpha$ such that $p \leq q$ and $\sigma(p) = \sigma(q)$.

PROOF. We prove the claim by induction on α .

Assume $\alpha = \beta + 1$. If $\beta \in E$, there is nothing to be proved. So assume $\beta \in \mathcal{O}$. By Lemma 2 we can find $q' \leq q \upharpoonright \beta$ and a $\mathcal{C}(\beta)$ -name a_p^β such that $\sigma(q') = \sigma(q \upharpoonright \beta) = \sigma(q)$ and

$$q' \Vdash_{\mathcal{P}_\beta} \text{the first coordinate of } q(\beta) \text{ is equal to } a_p^\beta.$$

By the induction hypothesis we can find a $p' \in \mathcal{Q}_\beta$ such that $p' \leq q'$ and $\sigma(p') = \sigma(q')$. Let

$$p = p' \cup \{(\beta, \langle a_p^\beta, A_p^\beta \rangle)\},$$

where A_p^β is a \mathcal{P}_β -name such that $\Vdash_{\mathcal{P}_\beta} A_p^\beta$ is the second coordinate of $q(\beta)$. Then $p \in \mathcal{D}_\alpha$, $p \leq q$ and $\sigma(p) = \sigma(q)$.

First assume cf $\alpha = \omega$. Let $\langle \alpha_n : n < \omega \rangle$ be a strictly increasing sequence of ordinals cofinal with α such that $\alpha_0 = 0$. By induction on $n < \omega$ we construct a sequence $\langle p_n : n < \omega \rangle$ of elements of \mathcal{P}_α such that $p_0 = q$ and

(1) $p_{n+1} \leq p_n$ and $\sigma(p_{n+1}) = \sigma(p_n)$,

(2) $p_{n+1} \restriction \alpha_{n+1} \in \mathcal{D}_{\alpha_{n+1}}$.

Assume p_n is defined. By the induction hypothesis we can find $p'_{n+1} \in \mathcal{D}_{\alpha_{n+1}}$ such that $p'_{n+1} \leq p_n \restriction \alpha_{n+1}$ and $\sigma(p'_{n+1}) = \sigma(p_n \restriction \alpha_{n+1})$. Let $p_{n+1} \in \mathcal{P}_\alpha$ be defined by $p_{n+1} \restriction \alpha_{n+1} = p'_{n+1}$ and $p_{n+1}(\beta) = p_n(\beta)$ for all $\beta \in [\alpha_{n+1}, \alpha)$. Then $p_{n+1} \leq p_n$, $\sigma(p_{n+1}) = \sigma(p_n)$, and $p_{n+1} \restriction \alpha_{n+1} \in \mathcal{D}_{\alpha_{n+1}}$.

Define $p \in \mathcal{P}_\alpha$ as follows. If $\beta \in E$, let $p(\beta) = q(\beta)$. So suppose $\beta \in O$, and let $n < \omega$ be such that $\beta \in [\alpha_n, \alpha_{n+1})$. Let a_p^β be a $\mathcal{C}(\beta)$ -name for the closure of

$$\cup \{a_{p_i}^\beta : n < i < \omega\},$$

and let A_p^β be a \mathcal{P}_β -name for $\cap \{A_{p_i}^\beta : n < i < \omega\}$. Let $p(\beta) = \langle a_p^\beta, A_p^\beta \rangle$. Then $p \in \mathcal{D}_\alpha$, $p \leq q$ and $\sigma(p) = \sigma(q)$.

Now assume cf $\alpha > \omega$. Since $\text{supp}(q)$ is countable, there is a $\gamma < \alpha$ such that $\text{supp}(q) \subseteq \gamma$. Using the induction hypothesis, we can find a $p' \in \mathcal{D}_\gamma$ such that $p' \leq q \restriction \gamma$ and $\sigma(p') = \sigma(q \restriction \gamma) = \sigma(q)$. Define $p \in \mathcal{P}_\alpha$ by $p \restriction \gamma = p'$ and $p(\beta) = \emptyset$ for $\beta \in [\gamma, \alpha) \cap E$ and $p(\beta) = \langle \emptyset, \omega_1 \rangle$ for $\beta \in [\gamma, \alpha) \cap O$. Then $p \in \mathcal{D}_\alpha$, $p \leq q$ and $\sigma(p) = \sigma(q)$. This proves the claim.

Suppose $p, q \in \mathcal{D}_\alpha$ are such that $p(\beta) = q(\beta)$ for every $\beta \in E$ and $\Vdash_{\mathcal{C}(\beta)} a_p^\beta = a_q^\beta$ for every $\beta \in O$. We claim that then p and q are compatible in \mathcal{P}_α . To see this let us define $p' \in \mathcal{D}_\alpha$ as follows. If $\beta \in E$, let $p'(\beta) = p(\beta) = q(\beta)$. If $\beta \in O$, we choose $p'(\beta)$ to satisfy $\Vdash_{\mathcal{P}_\beta} p'(\beta) = \langle a_p^\beta, A_p^\beta \cap A_q^\beta \rangle$. Then clearly $p' \in \mathcal{D}_\alpha$ and $p' \leq p, q$. Since there are only \aleph_1 $\mathcal{C}(\alpha)$ -names of countable closed subsets of ω_1 , this finishes the proof of Lemma 3.

Now we are ready to finish the proof of Theorem 1. Assume GCH holds. Let $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$ be the iteration defined above and let $\mathcal{P} = \mathcal{P}_{\omega_2}$. Then in $V^\mathcal{P}$, $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ holds. Working in $V^\mathcal{P}$, we define a finite support iteration $\langle \dot{\mathcal{Q}}_\xi : \xi \leq \omega_2 \rangle$ of c.c.c. posets of size $\leq \aleph_1$ à la Solovay and Tennenbaum [33], such that if $\dot{\mathcal{Q}} = \dot{\mathcal{Q}}_{\omega_2}$, then $V^{\mathcal{P} * \dot{\mathcal{Q}}}$ satisfies (i)–(iii) of Theorem 1.

Assume $\xi < \omega_2$ and that in $V^{\mathcal{P} * \dot{\mathcal{Q}}_\xi}$ we have a partition $[\omega_1]^2 = \dot{K}_0 \cup \dot{K}_1$ with no bad sets. Pick an even ordinal $\alpha < \omega_2$ such that $\dot{\mathcal{Q}}_\xi, \dot{K}_0, \dot{K}_1 \in V^{\mathcal{P}_\alpha}$. We have already remarked that $\mathcal{P}_{\alpha, \omega_2}$ is, in $V^{\mathcal{P}_\alpha}$ (equivalent to) a mixed iteration of length ω_2 of Cohen and Jensen posets. It begins by first introducing \aleph_1 Cohen reals and then adding a Jensen club. So by Lemma 1, the poset $\dot{\mathcal{S}}_\xi$ of all finite 0-homogeneous subsets of ω_1 which are separated by $C_{\dot{\mathcal{Q}}_{\alpha+1}}$ is a c.c.c. poset in $V^{\mathcal{P}_{\alpha+2} * \dot{\mathcal{Q}}_\xi}$. Hence, one condition $s_0 \in \dot{\mathcal{S}}_\xi$ forces that the generic object is uncountable. At the next step of the iteration we force with $\{s \in \dot{\mathcal{S}}_\xi : s \supseteq s_0\}$, which we again denote by $\dot{\mathcal{S}}_\xi$. We know that $\dot{\mathcal{Q}}_\xi * \dot{\mathcal{S}}_\xi$ is a c.c.c. poset in $V^{\mathcal{P}_{\alpha+2}}$, but we have to show that it remains c.c.c. after forcing with $\mathcal{P}_{\alpha+2, \omega_2}$. Let us prove the following more general fact. Let $\dot{\mathcal{Q}}$ be an arbitrary c.c.c. poset. Then $\dot{\mathcal{Q}}$ remains c.c.c. after forcing with $\mathcal{P} = \mathcal{P}_{\omega_2}$. Otherwise, pick a \mathcal{P} -name

$\langle \dot{q}_\gamma: \gamma < \omega_1 \rangle$ for an ω_1 -sequence of incompatible members of $\bar{\mathcal{Q}}$. As in the proof of Lemma 2, by induction on γ we construct a decreasing sequence $\langle p_\gamma: \gamma < \omega_1 \rangle$ of members of \mathcal{P} , a sequence $\langle r_\gamma: \gamma < \omega_1 \rangle$ of members of $\mathcal{C}_{E \times \omega_1}$, and a sequence $\langle q_\gamma: \gamma < \omega_1 \rangle$ of members of $\bar{\mathcal{Q}}$ such that

(1) $\sigma(p_\gamma) = \sigma(p_\delta)$ for $\gamma < \delta < \omega_1$,

(2) $p_\gamma \wedge r_\gamma \Vdash_{\mathcal{P}} \dot{q}_\gamma = q_\gamma$.

Pick an $A \in [\omega_1]^{\aleph_1}$ such that r_γ and r_δ are compatible, whenever $\gamma, \delta \in A$. Then it is easily checked that we have reached a contradiction since $\{q_\gamma: \gamma \in A\}$ is an uncountable antichain of $\bar{\mathcal{Q}}$.

Similarly, one defines posets for getting $\omega_1 \xrightarrow{*} (\omega_1)_{<\aleph_0}^2$ as well as the posets for getting MA_{\aleph_1} . This completes the proof of Theorem 1.

The following result (in ZFC) is an easy consequence of Lemma 1.

THEOREM 2. $\omega_1 \xrightarrow{*} (\text{closed } \alpha)_{<\aleph_0}^2$ for all $\alpha < \omega_1$.

PROOF. Let $[\omega_1]^2 = \bigcup_{\xi < \omega_1} J_\xi$ be a given disjoint partition such that $|J_\xi| < \aleph_0$ for all $\xi < \omega_1$. Let $\omega \leq \alpha < \omega_1$ be fixed. Let $\dot{S} \in V^{\mathcal{C}_{\omega_1} * \dot{E}}$ be the set of all finite subsets of ω_1 separated by $\dot{C}_{\dot{S}}$ such that $||s]^2 \cap J_\xi| \leq 1$ for all $\xi < \omega_1$. By Lemma 1, \dot{S} is a c.c.c. poset in $V^{\mathcal{C}_{\omega_1} * \dot{E}}$. So we can find an $s_0 \in \dot{S}$ such that $s_0 \Vdash_{\dot{S}} \bigcup \dot{C}_{\dot{S}}$ is stationary in ω_1 . Thus, in particular, we can find a $p_0 \in \mathcal{C}_{\omega_1} * \dot{E} * \dot{S}$ and a $(\mathcal{C}_{\omega_1} * \dot{E} * \dot{S})$ -name \dot{A} such that

$$p_0 \Vdash \dot{A} \text{ is a closed subset of } \omega_1 \text{ of type } \alpha + 1 \ \& \ \forall \xi < \omega_1 \ |[\dot{A}]^2 \cap J_\xi| \leq 1.$$

Pick a $(\mathcal{C}_{\omega_1} * \dot{E} * \dot{S})$ -name \dot{f} such that $p_0 \Vdash \dot{f}: \alpha + 1 \rightarrow \dot{A}$ is the unique isomorphism. Let $\langle \alpha_n: n < \omega \rangle$ be an enumeration of $\alpha + 1$. Now by induction on $n < \omega$ we define a decreasing sequence $\langle p_n: n < \omega \rangle$ of elements of $\mathcal{C}_{\omega_1} * \dot{E} * \dot{S}$ and a sequence $\langle \beta_n: n < \omega \rangle$ of ordinals $< \omega_1$ such that $p_{n+1} \Vdash \dot{f}(\alpha_n) = \beta_n$, making sure that $B = \{\beta_n: n < \omega\}$ is a closed subset of ω_1 . Then $\text{tp } B = \alpha + 1$ and $\forall \xi < \omega_1 \ | [B]^2 \cap J_\xi| \leq 1$. This completes the proof.

REMARKS. (1) The closed unbounded set poset was defined and first used in building c.c.c. posets in the extension by Jensen [8]. The fact that an elementary chain of submodels is useful in proving the c.c.c. property of posets with separated conditions was first realized by Shelah [1, 3]. The use of the Cohen generic reals in building conditions in σ -closed posets was first made explicit by Avraham [2]. The first mixed iteration of Cohen posets and σ -closed posets was defined by Mitchell [26]. It is clear that if we want only to preserve \aleph_1 , then in the above mixed iteration the Jensen posets can be replaced by any σ -closed poset. If we want the iteration to have the \aleph_2 -c.c., the σ -closed posets must satisfy one of the standard strong \aleph_2 -chain conditions.

(2) The posets involved in Lemma 1 can also be iterated in a countable support iteration $\langle \mathcal{T}_\alpha: \alpha \leq \omega_2 \rangle$. A Laver type argument shows that \mathcal{T}_{ω_2} preserves \aleph_1 [5, 25]. Using GCH, one then shows that \mathcal{T}_{ω_2} satisfies the \aleph_2 -c.c.

(3) If we are not interested in the exact equiconsistency result, we could use the Proper Forcing Axiom (PFA; [6, 7, 31]) in showing that (i)–(iii) of Theorem 1 are

consistent. Namely, in this case, in Lemma 1, we can disregard \mathcal{Q} and \mathcal{C}_{ω_1} and directly show by the same proof that \mathfrak{S}_0 and \mathfrak{S}_1 are c.c.c. posets in $V^\mathfrak{E}$. To build a condition which will meet all the \mathcal{U}_F 's, we need only use MA_{\aleph_1} , a consequence of PFA.

(4) It is clear that the proof of Lemma 1 also shows that each (finite) power of the poset \mathfrak{S}_1 satisfies the c.c.c. Hence the model of Theorem 1 can also satisfy the following partition property of ω_1 stronger than $\omega_1 \rightarrow^* (\omega_1)_{<\aleph_0}^2$:

If $[\omega_1]^2 = \bigcup_{i \in I} K_i$ is a disjoint partition where each K_i is finite, then there is a decomposition $\omega_1 = \bigcup_{n < \omega} A_n$ such that

$$\forall n < \omega \forall i \in I | [A_n]^2 \cap K_i | \leq 1.$$

2. This section begins with a discussion of the partition relation $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ and ends with the applications mentioned in the introduction.

For $G \subseteq [\omega_1]^2$, $\text{Chr}(G)$ denotes the chromatic number of G and equals the minimal cardinal κ for which there is a partition $\omega_1 = \bigcup_{\xi < \kappa} A_\xi$ such that $[A_\xi]^2 \cap G = \emptyset$ for all $\xi < \kappa$.

THEOREM 3. *Assume MA_{\aleph_1} and $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Then for every $G \subseteq [\omega_1]^2$ either $\text{Chr}(G) \leq \aleph_0$, or else there is an $A \in [\omega_1]^{\aleph_1}$ and a family \mathfrak{B} of \aleph_1 disjoint finite subsets of ω such that $(\{\alpha\} \otimes F) \cap G \neq \emptyset$ for all $\alpha \in A$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.*

PROOF. Given $G \subseteq [\omega_1]^2$, let \mathcal{P} be the set of all finite $p \subseteq \omega_1$ such that $[p]^2 \cap G = \emptyset$. The ordering on \mathcal{P} is \supseteq .

If \mathcal{P} is a c.c.c. poset, then by MA_{\aleph_1} , \mathcal{P} is σ -centered, so $\text{Chr}(G) \leq \aleph_0$.

Hence, we may assume \mathcal{P} is not a c.c.c. poset. Let $\{p_\alpha: \alpha < \omega_1\}$ be an uncountable antichain of \mathcal{P} . A standard Δ -system argument shows we may assume the p_α 's are disjoint, strictly increasing and of the same cardinality n ($1 \leq n < \omega$). Let $\langle p_\alpha(i): i < n \rangle$ be the strictly increasing enumeration of p_α , ($\alpha < \omega_1$). For each $\alpha < \beta < \omega_1$, there exist $i, j < n$ such that $\{p_\alpha(i), p_\beta(j)\} \in G$. This gives a coloring of $[\omega_1]^2$ into n^2 colors. Now an easy application of $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ completes the proof of Theorem 3.

A consequence of Theorem 3 is that, in the model of §1,

$$\omega_1 \rightarrow (\text{stationary}, (\omega_1; \text{fin } \omega_1))^2$$

holds. However, an examination of the proof of Theorem 1 shows that, in fact, in this model, the stronger relation

$$\omega_1 \rightarrow (\text{stationary}, (\text{stationary}; \text{fin } \omega_1))^2$$

holds. Let us also mention the following strengthening (*) of $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ in a dual direction. The consistency of this strengthening will appear in a later paper.

(*) For every partition $[\omega_1]^2 = K_0 \cup K_1$ either there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or else there exist $\langle A_n: n < \omega \rangle$ and $\langle \mathfrak{B}_n: n < \omega \rangle$ such that:

- (i) $\omega_1 \setminus \bigcup_{n < \omega} A_n$ is countable;
- (ii) \mathfrak{B}_n is a family of \aleph_1 disjoint finite subsets of ω_1 ;
- (iii) $(\{\alpha\} \otimes F) \cap K_1 \neq \emptyset$ for all $\alpha \in A_n$ and $F \in \mathfrak{B}_n$ with $\alpha < \min F$.

Let us note that it is not possible to strengthen $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ at the same time in both the direction of Theorem 3 and that of (*), i.e., there is a partition $[\omega_1]^2 = K_0 \cup K_1$ with no stationary 0-homogeneous sets, but ω_1 is not a countable union of bad sets. A proof of this simple fact will also appear elsewhere.

THEOREM 4. *Assume MA_{\aleph_1} and $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Then for every partition $[\omega_1]^2 = K_0 \cup K_1$ either there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or else for every $\alpha < \omega_1$ there are $B, C \subseteq \omega_1$ such that $\text{tp } B = \alpha$, $|C| = \aleph_1$ and $[B]^2 \cup (B \otimes C) \subseteq K_1$.*

In particular we have the following consequence mentioned in the introduction.

THEOREM 5. *Assume MA_{\aleph_1} . Then $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ implies $\omega_1 \rightarrow (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$.*

PROOF OF THEOREM 4: Let $[\omega_1]^2 = K_0 \cup K_1$ be a given partition, and assume $[A]^2 \not\subseteq K_0$ for all $A \in [\omega_1]^{\aleph_1}$. For each $\alpha < \omega_1$ we shall construct $B, C \subseteq \omega_1$ such that $\text{tp } B = \omega^\alpha$, $|C| = \aleph_1$ and $[B]^2 \cup (B \otimes C) \subseteq K_1$. First we need some technical definitions and facts.

For each $1 \leq \alpha < \omega_1$ we fix a nondecreasing sequence $\langle \alpha(n) : n < \omega \rangle$ of smaller ordinals such that $\omega^\alpha = \sum_{n < \omega} \omega^{\alpha(n)}$, and if $\alpha > 1$, then $\alpha(0) \geq 1$. Also for every set $B \subseteq \omega_1$ of type ω^α we fix a decomposition $B = \bigcup_{n < \omega} B(n)$ such that

$$B(0) < \dots < B(n) < \dots \quad \text{and} \quad \text{tp } B(n) = \omega^{\alpha(n)}.$$

Let \mathcal{V} be a fixed nonprincipal ultrafilter on ω_1 . By induction on $1 \leq \alpha < \omega_1$ we define a nonprincipal ultrafilter $\mathcal{U}_\alpha(B)$ on every set $B \subseteq \omega_1$ of type ω^α . If $\alpha = 1$, then the isomorphism of ω and B induces $\mathcal{U}_\alpha(B)$. So now assume $1 < \alpha < \omega_1$ and define

$$D \in \mathcal{U}_\alpha(B) \quad \text{iff} \quad \{n < \omega : D \cap B(n) \in \mathcal{U}_{\alpha(n)}(B(n))\} \in \mathcal{V}.$$

By induction on α it easily follows that $\text{tp } D = \omega^\alpha$ for every $D \in \mathcal{U}_\alpha(B)$. The following lemma is due to Hajnal [24, p. 1031]. For the sake of completeness we sketch the proof.

CLAIM 4. Let $1 \leq \alpha < \omega_1$ and let $B \subseteq \omega_1$ have type ω^α . Let $\langle D_\xi : \xi < \omega_1 \rangle$ be a sequence of elements of $\mathcal{U}_\alpha(B)$. Then there exists a $D \subseteq B$, with $\text{tp } D = \omega^\alpha$ such that $D \setminus D_\xi$ is a bounded subset of D for every $\xi < \omega_1$.

PROOF. The proof is by induction on α . The case $\alpha = 1$ is a well-known consequence of MA_{\aleph_1} . So let $1 < \alpha < \omega_1$. By the induction hypothesis, for each $n < \omega$, there is an $E_n \subseteq B(n)$ of type $\omega^{\alpha(n)}$ such that $E_n \setminus D_\xi$ is bounded in E_n for all $\xi < \omega_1$ with the property

$$n \in N_\xi = \{m < \omega : D_\xi \cap B(m) \in \mathcal{U}_{\alpha(m)}(B(m))\}.$$

Now for each $\xi < \omega_1$ we fix $f_\xi \in {}^\omega \omega$ with the property that for every $n \in N_\xi$, the $f_\xi(n)$ -end-section of E_n is a subset of D_ξ . Let $N \subseteq \omega$ be an infinite set almost included in each N_ξ , and let $f \in {}^\omega \omega$ eventually dominate each f_ξ . For $n \in N$, let D_n be the $f(n)$ -end-section of E_n . Let $D = \bigcup_{n \in N} D_n$. Then D is as required.

Now we are ready for the proof of Theorem 4. By induction on $\alpha < \omega_1$, for each $A \in [\omega_1]^{\aleph_1}$ we shall construct $B, C \subseteq A$ such that $\text{tp } B = \omega^\alpha$, $|C| = \aleph_1$ and $[B]^2 \cup (B \otimes C) \subseteq K_1$. Since the case $\alpha = 0$ is trivial we assume $1 \leq \alpha < \omega_1$. Let $A \in [\omega_1]^{\aleph_1}$

be given. By $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ there is an $A_0 \in [A]^{\aleph_1}$ and a family \mathfrak{B} of \aleph_1 disjoint finite subsets of A such that $(\{\beta\} \otimes F) \cap K_1 \neq \emptyset$ for all $\alpha \in A_0$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.

Using the induction hypothesis we recursively construct sets $B_n, C_n \subseteq A_0$ ($n < \omega$) such that:

- (1) $\text{tp } B_n = \omega^{\alpha(n)} \& |C_n| = \aleph_1$;
- (2) $B_n < B_{n+1} \& C_n \supseteq C_{n+1}$;
- (3) $B_{n+1} \subseteq C_n$;
- (4) $[B_n]^2 \cup (B_n \otimes C_n) \subseteq K_1$.

Let $B = \bigcup_{n < \omega} B_n$. Then $\text{tp } B = \omega^\alpha$ and $[B]^2 \subseteq K_1$. Pick $F \in \mathfrak{B}$ such that $\sup B \leq \min F$. By the assumptions on A_0 and \mathfrak{B} , we have that $B \subseteq \bigcup_{\gamma \in F} K_1(\gamma)$. Hence for some $\gamma = \gamma(F) \in F$, we have $K_1(\gamma) \cap B \in \mathcal{U}_\alpha(B)$. By Claim 4 there is a $D \subseteq B$ with $\text{tp } D = \omega^\alpha$ such that $D \setminus K_1(\gamma(F))$ is bounded in D for every $F \in \mathfrak{B}$, with $\sup B \leq \min F$. Thus, for some uncountable $\mathfrak{B}_0 \subseteq \mathfrak{B}$ and $\delta \in D$ we have $D \setminus \delta \subseteq K_1(\gamma(F))$ and $\sup B \leq \min F$ for all $F \in \mathfrak{B}_0$. Let $B^* = D \setminus \delta$ and $C^* = \{\gamma(F) : F \in \mathfrak{B}_0\}$. Then $B^*, C^* \subseteq A$, $\text{tp } B^* = \omega^\alpha$, $|C^*| = \aleph_1$, and $[B^*]^2 \cup (B^* \otimes C^*) \subseteq K_1$. This completes the proof.

Let us now consider the following combinatorial principle introduced by Fred Galvin:

(**) There are ideals $\mathcal{G}, \mathcal{F} \subseteq \mathcal{P}(\omega_1)$ such that:

- (i) $\mathcal{G} \cap \mathcal{F} = [\omega_1]^{<\aleph_0}$;
- (ii) $\mathcal{G} \vee \mathcal{F} = \{A \cup B : A \in \mathcal{G} \& B \in \mathcal{F}\} = [\omega_1]^{<\aleph_0}$;
- (iii) $\forall A \in [\omega_1]^{\aleph_1} ([A]^{\aleph_0} \cap \mathcal{G} \neq \emptyset \& [A]^{\aleph_0} \cap \mathcal{F} \neq \emptyset)$.

Galvin proved that \clubsuit implies (**) and that (**) has some topological applications [15, Theorem 4]. He also asked for the consistency of $\neg(**)$. The next result shows that $\neg(**)$ is consistent.

THEOREM 6. $\omega_1 \rightarrow (\omega_1; \text{fin } \omega_1)_2^2$ implies $\neg(**)$.

PROOF. Let \mathcal{G} and \mathcal{F} be ideals satisfying (i) and (ii) of (**). For each $\alpha < \omega_1$ we can find disjoint $A_\alpha \in \mathcal{G}$ and $B_\alpha \in \mathcal{F}$ such that $A_\alpha \cup B_\alpha = \alpha$. Define $[\omega_1]^2 = K_0 \cup K_1$ by

$$\{\beta, \alpha\} < \in K_0 \quad \text{iff} \quad \beta \in A_\alpha.$$

Since $\omega_1 \rightarrow (\omega_1; \text{fin } \omega_1)_2^2$ holds, we consider the following two cases:

Case I. There is an $A \in [\omega_1]^{\aleph_1}$ and a family \mathfrak{B} of \aleph_1 disjoint finite subsets of ω_1 such that $(\{\alpha\} \otimes F) \cap K_0 \neq \emptyset$ for all $\alpha \in A$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.

For $F \in \mathfrak{B}$ we define $A(F) = \bigcup \{A_\alpha : \alpha \in F\}$. Then $A(F) \in \mathcal{G}$ and $A \cap \min F \subseteq A(F)$ for each $F \in \mathfrak{B}$. Hence $[A]^{\aleph_0} \subseteq \mathcal{G}$, contradicting the conjunction of (i) and (iii). This shows that (**) fails in this case.

Case II. There is an $A \in [\omega_1]^{\aleph_1}$ and a family \mathfrak{B} of \aleph_1 disjoint finite subsets of ω_1 such that $(\{\alpha\} \otimes F) \cap K_1 \neq \emptyset$ for all $\alpha \in A$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.

Proceeding as in Case I we show that here $[A]^{\aleph_0} \subseteq \mathcal{F}$, which again contradicts (**). This completes the proof.

The remainder of this section is devoted to the topological applications mentioned in the introduction.

THEOREM 7. Assume $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Let X be a topological space with no uncountable discrete subspaces. Let \mathcal{U} be a family of open subsets of X such that $\bigcup \mathcal{U} = X$. Then there is a countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $X = \bigcup \{\bar{U} : U \in \mathcal{U}_0\}$.

PROOF. Assume by way of contradiction that, for every countable $\mathcal{U}_0 \subseteq \mathcal{U}$, $X \neq \bigcup \{\bar{U} : U \in \mathcal{U}_0\}$. Then by induction on $\alpha < \omega_1$, we can easily construct sequences $\langle U_\alpha : \alpha < \omega_1 \rangle$ and $\langle x_\alpha : \alpha < \omega_1 \rangle$ of members of \mathcal{U} and X , respectively, such that

- (1) $x_\alpha \in U_\alpha$,
- (2) $x_\alpha \notin \bigcup \{\bar{U}_\beta : \beta < \alpha\}$.

Define $[\omega_1]^2 = K_0 \cup K_1$ by

$$\{\beta, \alpha\} < \in K_0 \quad \text{iff} \quad x_\beta \notin U_\alpha.$$

Since $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ holds, we consider the following two cases:

Case I. There is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$. Then for every $\alpha \in A$, $U_\alpha \cap \{x_\beta : \beta \in A\} = \{x_\alpha\}$. Hence $\{x_\alpha : \alpha \in A\}$ is an uncountable discrete subspace of X , a contradiction.

Case II. There is an $A \in [\omega_1]^{\aleph_1}$ and a family \mathfrak{B} of \aleph_1 disjoint subsets of ω_1 such that $(\{\alpha\} \otimes F) \cap K_1 \neq \emptyset$ for all $\alpha \in A$ and $F \in \mathfrak{B}$ with $\alpha < \min F$. For $F \in \mathfrak{B}$ we define

$$U(F) = \bigcup_{\gamma \in F} U_\gamma.$$

Then for each $F \in \mathfrak{B}$,

$$\{x_\alpha : \alpha \in A \cap \min F\} \subseteq U(F).$$

Choose inductively an $A_0 \in [A]^{\aleph_1}$ and, for each $\alpha \in A_0$, an $F_\alpha \in \mathfrak{B}$ such that if $\beta < \alpha$ are in A_0 , then

$$\max F_\beta < \beta < \min F_\alpha \leq \max F_\alpha < \alpha.$$

Then by (1) and (2), for each $\alpha \in A_0$,

$$(U_\alpha \setminus \overline{U(F_\alpha)}) \cap \{x_\beta : \beta \in A_0\} = \{x_\alpha\}.$$

Hence $\{x_\alpha : \alpha \in A_0\}$ is an uncountable discrete subspace of X , a contradiction. This completes the proof.

THEOREM 8. Assume $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Then every regular topological space with no uncountable discrete subspace is hereditarily Lindelöf.

PROOF. Clearly it suffices to show that X is Lindelöf. So let \mathcal{U} be a family of open sets such that $\bigcup \mathcal{U} = X$. Since X is a regular space, there is a family \mathcal{V} of open subsets of X such that $\bigcup \mathcal{V} = X$, and such that for every $W \in \mathcal{V}$ there is a $U(W) \in \mathcal{U}$ such that $U(W) \supseteq \bar{W}$. By Theorem 7 there is a countable $\mathcal{V}_0 \subseteq \mathcal{V}$ such that

$$X = \bigcup \{\bar{W} : W \in \mathcal{V}_0\}.$$

Hence $\mathcal{U}_0 = \{U(W) : W \in \mathcal{V}_0\}$ is a countable subfamily of \mathcal{U} such that $\bigcup \mathcal{U}_0 = X$. This completes the proof.

COROLLARY 9. Assume $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Then every regular hereditarily separable topological space is hereditarily Lindelöf.

THEOREM 10. Assume $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Let X be a Hausdorff space with no uncountable discrete subspaces. Then every point of X is the intersection of countably many open subsets of X .

PROOF. Fix $x \in X$. Since X is a Hausdorff space, for every $y \in X \setminus \{x\}$ there is an open set U_y such that $y \in U_y$ and $x \notin \overline{U_y}$. By Theorem 7, applied to the space $X \setminus \{x\}$, there is a countable $Y \subseteq X \setminus \{x\}$ such that $X \setminus \{x\} = \bigcup \{\overline{U_y} : y \in Y\}$. This shows that $\{x\}$ is a G_δ subset of X .

The following theorem is a simple consequence of Theorem 10 using a result of [18]. However, since the result we need is a relatively simple application of $(2^{\aleph_0})^+ \rightarrow (\aleph_1)^2_{\aleph_0}$, we shall give some details.

THEOREM 11. Assume $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Then every Hausdorff space with no uncountable discrete subspaces has cardinality $\leq 2^{\aleph_0}$.

PROOF. Assume by way of contradiction that X is a Hausdorff space of cardinality $> 2^{\aleph_0}$ with no uncountable discrete subspaces.

Let $<$ be a well-ordering of X . By Theorem 10, for each $x \in X$ we can fix a family $\{U_x^n : n < \omega\}$ of open subsets of X such that $\{x\} = \bigcap_{n < \omega} U_x^n$. For $m, n < \omega$ and $\{x, y\} < [X]^2$ we let

$$\{x, y\} \in K_{m,n} \quad \text{iff} \quad x \notin U_y^n \text{ \& } y \notin U_x^m.$$

Clearly, $[X]^2 = \bigcup_{m,n < \omega} K_{m,n}$. By $(2^{\aleph_0})^+ \rightarrow (\aleph_1)^2_{\aleph_0}$, there are $m', n' < \omega$ and $D \in [X]^{\aleph_1}$ such that $[D]^2 \subseteq K_{m',n'}$. For $x \in D$, let $W_x = U_x^{m'} \cap U_x^{n'}$. Then for each $x \in D$, $W_x \cap D = \{x\}$. Hence, D is a discrete subspace of X , a contradiction. This completes the proof.

We conclude the paper with a remark on the following partition relation (it is dual to $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2$), denoted by

$$\omega_1 \rightarrow (\omega_1, (\text{fin } \omega_1; \omega_1))^2.$$

This relation means: For every partition $[\omega_1]^2 = K_0 \cup K_1$ either

- (1) there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or
- (2) there is a family \mathcal{Q} of \aleph_1 disjoint finite subsets of ω_1 and a set $B \in [\omega_1]^{\aleph_1}$ such that $(F \otimes \{\beta\}) \cap K_1 \neq \emptyset$ for all $F \in \mathcal{Q}$ and $\beta \in B$ with $\max F < \beta$.

The consistency of $\omega_1 \rightarrow (\omega_1, (\text{fin } \omega_1; \omega_1))^2$ is an open problem. It is easily seen that $\omega_1 \rightarrow (\omega_1, (\text{fin } \omega_1; \omega_1))^2$ implies the dual statement of Theorem 8, i.e., that every regular space with no uncountable discrete subspaces is hereditarily separable.

The reader interested in the role of MA_{\aleph_1} in the problems we have considered here can find some information in [4].

REFERENCES

1. U. Avraham and S. Shelah, *Martin's axiom does not imply that every two \aleph_1 -dense sets of reals are isomorphic*, Israel J. Math. **38** (1981), 161–176.
2. ———, *Isomorphism types of Aronszajn trees* (to appear).
3. U. Avraham, M. Rubin and S. Shelah, *On the consistency of some partition theorems for continuous colorings, and the structure of \aleph_1 -dense real order types* (to appear).

4. U. Avraham and S. Todorčević, *Martin's axiom and first countable S and L spaces*, Handbook of the Set-Theoretic Topology (to appear).
5. J. Baumgartner, *Iterated forcing* (to appear).
6. ———, *Application of the proper forcing axiom*, Handbook of the Set-Theoretic Topology (to appear).
7. K. J. Devlin, *The Yorkshireman's guide to proper forcing* (to appear).
8. K. J. Devlin and H. Johnsråten, *The Souslin problem*, Lecture Notes in Math., vol. 405, Springer-Verlag, Berlin and New York, 1974.
9. B. Dushnik and E. W. Miller, *Partially ordered sets*, Amer. J. Math. **63** (1941), 600–610.
10. B. A. Efimov, *On the cardinality of Hausdorff spaces*, Dokl. Akad. Nauk SSSR **164** (1965), 967–970. (Russian)
11. P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489.
12. P. Erdős and A. Hajnal, *Unsolved problems in set theory*, Axiomatic Set Theory, Proc. Sympos. Pure Math., vol. 13, Part I, Amer. Math. Soc., Providence, R.I., 1971, pp. 14–48.
13. ———, *Unsolved and solved problems in set theory* (Proceedings of the Tarski Symposium), Proc. Sympos. Pure Math., vol. 25, Amer. Math. Soc., Providence, R.I., 1974, pp. 267–287.
14. V. Fedorčuk, *On the cardinality of hereditarily separable compact Hausdorff spaces*, Soviet Math. Dokl. **16** (1975), 651–655.
15. F. Galvin, *On Gruenhage's generalization of first countable spaces. II*, Notices Amer. Math. Soc. **24** (1977), A–257.
16. J. de Groot, *Discrete subspaces of Hausdorff spaces*, Bull. Acad. Polon. Sci. **13** (1965), 537–544.
17. A. Hajnal, *Some results and problems in set theory*, Acta Math. Acad. Sci. Hungar. **11** (1960), 277–298.
18. A. Hajnal and I. Juhász, *Discrete subspaces of topological spaces*, Nederl. Akad. Wetensch. Indag. Math. **29** (1967), 343–356.
19. ———, *A consistency result concerning hereditarily α -separable spaces*, Nederl. Akad. Wetensch. Indag. Math. **81** (1973), 301–307.
20. J. Isbell, *Remarks on spaces of large cardinal numbers*, Czechoslovak Math. J. **14** (89) (1964), 383–385.
21. T. J. Jech, *Set theory*, Academic Press, New York, 1978.
22. I. Juhász, *A survey of S and L spaces*, Topology Colloq. Math. Soc. János Bolyai, vol. 23, North-Holland, Amsterdam, 1980, pp. 674–688.
23. K. Kunen, *Set theory*, North-Holland, Amsterdam, 1980.
24. R. Laver, *Partition relations for uncountable cardinals $\leq 2^{\aleph_0}$* , Infinite and Finite sets, Vol. II, Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975, pp. 1029–1042.
25. ———, *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), 151–169.
26. W. J. Mitchell, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic **5** (1972), 21–46.
27. F. P. Ramsey, *On the problem of formal logic*, Proc. London Math. Soc. (2) **30** (1930), 264–286.
28. J. Roitman, *Basic S and L* , Handbook of the Set-Theoretic Topology (to appear).
29. M. E. Rudin, *A normal hereditarily separable non-Lindelöf space*, Illinois J. Math. **10** (1972), 621–626.
30. ———, *S and L spaces*, Surveys in General Topology, Academic Press, 1980, pp. 431–444.
31. S. Shelah, *Proper forcing*, Lectures Notes in Math., vol. 940, Springer-Verlag, Berlin, Heidelberg and New York, 1982.
32. W. Sierpiński, *Sur un problème de la théorie des relations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **2** (1933), 285–287.
33. R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. (2) **94** (1971), 201–245.
34. S. Todorčević, *On the S -space problem*, Abstracts Amer. Math. Soc. **2** (1981), no. 4, A–394.
35. ———, $\omega_1 \rightarrow (\omega_1, \omega + 2)^2$ is consistent, Abstracts Amer. Math. Soc. **2** (1981), no. 5, A–462.
36. ———, *On the cardinality of Hausdorff spaces*, Abstracts Amer. Math. Soc. **2** (1981), no. 6, A–529.
37. N. H. Williams, *Combinatorial set theory*, North-Holland, Amsterdam, 1977.